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PP-wave string interactions from n-point correlators of BMN operators

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Abstract: BMN operators are characterized by the fact that they have infinite R-charge and finite anomalous dimension in the BMN double scaling limit. Using this fact, we show that the BMN operators close under operator product expansion and form a sector in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We then identify short-distance limits of general BMN n-point correlators, and show how they correspond to the pp-wave string interactions. We also discuss instantons in the light of the pp-wave/SYM correspondence.

Keywords: String Field Theory, Penrose limit and pp-wave background.

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Contents

1. Introduction

Recently Berenstein, Maldacena and Nastase (BMN) [1] put forward a remarkable proposal of a correspondence between certain operators in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory (SYM) and massive states in string theory in a pp-wave background [2]

$$\frac{1}{\sqrt{J}N^{J/2+1}}\,\mathrm{Tr}\,Z^J \longleftrightarrow |0,p^+\rangle\,, \tag{1.1}$$

$$\frac{1}{\sqrt{J}N^{J/2+1}}\sum_{l=0}^{J}\mathrm{Tr}\left[\phi_3 Z^l \phi_4 Z^{J-l}\right]e^{\frac{2\pi i n l}{J}} \longleftrightarrow a_n^{7\dagger}a_{-n}^{8\dagger}|0,p^+\rangle\,. \tag{1.2}$$

It is often said that the BMN operators form a *sector* in SYM in the double scaling limit

$$N\to\infty\,,\qquad J\sim\sqrt{N}\qquad \text{with } g_{\mathrm{YM}} \text{ fixed.} \tag{1.3}$$

In this paper, we will give a more precise meaning to this statement using the operator product expansion (OPE). We will argue that, in the double scaling limit, the OPE of BMN operators does not give rise to non-BMN operators. We will use this *short OPE* to analyse certain short distance limits of general n-point correlators of the BMN operators, and find a precise correspondence with the structure of the string interactions in the pp-wave background.

The BMN correspondence holds in the double scaling limit (1.3). In this limit the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ is infinite and perturbative calculations in gauge theory are not under control. BMN instead concentrated on a class of 'near-BPS' operators with large R-charge J, e.g. as in (1.2). For these operators the coupling is effectively

$$\lambda' = \frac{g_{\mathrm{YM}}^2 N}{J^2} = \frac{1}{(\mu p^+\alpha')^2}\,, \tag{1.4}$$

which is finite in the large N limit (1.3) and can be taken small. The anomalous dimensions δ of the BMN operators are finite in the limit (1.3), and are related to the masses of the corresponding string states via[1]

$$\Delta - J = \frac{H_{\mathrm{lc}}}{\mu}\,, \tag{1.5}$$

where H_{lc} is the lightcone string hamiltonian and μ is the scale of the pp-wave metric. Written in terms of gauge theory parameters, the string theory gives a prediction for the conformal dimension of the BMN operators

$$(\Delta - J)_n = \sqrt{1 + \frac{g_{\mathrm{YM}}^2 N n^2}{J^2}}\,. \tag{1.6}$$

Since all the states in the perturbative spectrum of the string theory are already accounted for by the full set of BMN operators, it is a central part of the BMN proposal that the anomalous dimensions of the other (non-BMN) operators become infinite[2] in this limit (1.3), and hence the non-BMN operators play no rôle in the perturbative pp-wave/SYM correspondence.

The field theory side of the pp-wave/SYM correspondence was recently discussed in [3]–[6]. Non-planar diagrams in the BMN limit (1.3) were first studied in [3] and in [4] and were found to be important and governed by J^4/N^2. It follows from the double scaling limit (1.3) that in addition to λ' defined in (1.4), there is a second dimensionless constant

$$g_2 := \frac{J^2}{N} = 4\pi g_s (\mu p^+ \alpha')^2\,, \tag{1.7}$$

which plays the rôle of the genus counting parameter for the SYM Feynman diagrams [3, 4]. Anomalous dimensions were computed in [3, 4, 5, 7]. It was proposed in [4] that the coefficient of the three-point function of BMN operators in SYM is related to the three-string interactions in the pp-wave background. Planar three-point functions of BMN operators in free field theory were calculated in [4] and in the first nontrivial order of λ' in [6]. The proposal of [4] states that the matrix element of the lightcone hamiltonian is related to the coefficient C_{ijk} of the three-point function in field theory via

$$\langle i|P^-|j,k\rangle = \mu(\Delta_i - \Delta_j - \Delta_k) C_{ijk} \tag{1.8}$$

in the leading order in λ'. Checks of this in the free field limit were performed in [4, 8, 9]. See [10, 11] for further aspects about string interactions in pp-wave background. Another form of this proposal (which is insensitive to the prefactor) [12] relates the ratio of the three-string interactions with those of the field theory three-point function coefficients

$$\frac{\langle\Phi_1|\langle\Phi_2|\langle\Phi_3|V\rangle}{\langle 0_1|\langle 0_2|\langle 0_3|V\rangle} = \frac{C_{123}}{C_{123}^{(\mathrm{vac})}}\,. \tag{1.9}$$

[1] The BMN operator on the l.h.s. of (1.2) has two insertions of ϕ fields and its anomalous dimension is $\delta = \Delta - J - 2$. We will adopt the complex scalar field notation $Z = \phi_1 + i\phi_2$, $\Phi = \phi_3 + i\phi_4$ and $\Psi = \phi_5 + i\phi_6$. "Nonholomorphic" BMN operators, e.g. of the form $\sum_l \mathrm{Tr}[\Phi Z^l \bar{\Psi} Z^{J-l}] e^{\frac{2\pi i n l}{J}}$, are also allowed. What is not allowed are the operators with insertions of Φ and $\bar{\Phi}$ or Ψ and $\bar{\Psi}$ at the same time.

[2] Indeed it appears so in perturbation theory.

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Here the left hand side is the normalized 3-string interaction in the string field theory formalism [8], and $|V\rangle$ is the lightcone three-string vertex. In [6] the field theory results for the three-point function and the corresponding string theory prediction were derived and found to be in precise agreement, thus confirming (1.9) up to and including the order λ' corrections.

The next important problem to understand is how the higher point SYM correlation functions manifest themselves on the string theory side of the correspondence. This issue will be addressed in the present paper. The plan of the paper is as follow. In section 2, we show that the OPE of the BMN operators is closed in the double scaling limit. We also show that this short OPE has a very natural interpretation in string theory. Using this short OPE, we establish and extend in section 3 the correspondence between SYM correlators and pp-wave string interactions. We show that in a certain short distance limit involving a hierarchy of multi-pinchings, generic BMN correlators reduce to expressions written in terms of the three-point function coefficients and the anomalous dimensions. These expressions have a form that corresponds precisely to tree level string interactions in pp-wave background.[3] We also briefly discuss how loop corrected string interactions can be extracted from the BMN correlators. In section 4, we analyze instanton contributions to two- and three-point functions of BMN operators. We use a simple argument based on counting of fermion zero modes in the instanton background to show that two-point functions are protected from instanton corrections. This is consistent with the apparent absence of D-instanton solutions in pp-wave background. However, for generic three- and higher-point BMN correlators, our simple argument is not sufficient since fermion zero modes can be saturated in this case and one would require a detailed calculation to determine if the instanton corrections to the three-point functions are present.

Other relevant aspects of the correspondence have been studied in [14], where it was emphasised that the worldsheet model is exactly solvable in the lightcone gauge. Questions of holographic relation in the pp-wave context were considered in [15].

2. Short OPE of BMN operators

We first briefly recall the structure of operator product expansion (OPE). In a general QFT, the OPE is the statement that, in the short distance limit, the product of two local operators can be expressed in terms of a sum over local operators in the theory

$$\mathcal{O}_I(0)\mathcal{O}_J(x) = \sum_K c_{IJ}^K(x)\mathcal{O}_K(0)\,. \tag{2.1}$$

This is an operator relation. For generic correlators $\langle \mathcal{O}_I(0)\mathcal{O}_J(x)\prod_i A_i(y_i)\rangle$, the relation (2.1) holds only when $|x| \ll |y_i|$. For conformally invariant unitary theories, one can always choose to work with a basis of operators which do not mix with each other and have

[3] Very recently, after the first version of this paper appeared, it was argued in [13] that four- and higher-point functions are ill-defined in the BMN limit (1.3). We will point out in section 3 that our approach based on the OPE requires a specific order of limits: first we take the short-distance (pinching) limit, and then we take the BMN limit. In this case it follows that the inconsistencies mentioned in [13, sections 2.2 and 2.3] are not present.

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definite conformal dimensions. If the operators are also conformal primary operators, then conformal invariance of the theory implies that the two-point and three-point functions can be written in the form

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\rangle = \frac{\delta_{IJ}}{(4\pi^2 x_{12}^2)^{\Delta_I}}\,, \tag{2.2}$$

$$\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_3}(x_3)\rangle = \frac{C_{I_1 I_2 I_3}}{(4\pi^2 x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(4\pi^2 x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(4\pi^2 x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}\,, \tag{2.3}$$

where $x_{IJ}^2 := (x_I - x_J)^2$. The OPE takes then a simple form

$$\mathcal{O}_I(0)\mathcal{O}_J(x) = \sum_K \frac{C_{IJK}}{|x|^{\Delta_I+\Delta_J-\Delta_K}}\,\mathcal{O}_K(0)\,. \tag{2.4}$$

We will now consider the OPE of two BMN operators. It is known [3, 4] that the original single-trace BMN operators (for example, the l.h.s. of (1.2)) mix at the nonplanar level among themselves and with multi-trace operators [16, 13, 17]. Hence the original BMN operators do not have well defined conformal dimensions, and one has to define a new basis where the operators do not mix [3, 13]. This redefinition has to be implemented to all orders in λ' and g_2. Two different choices of these re-diagonalized bases were recently considered in [13, 18].

At present we don't know why (and if) the relevant SYM operators should be conformal primaries. However it has been shown recently that the three-point functions constructed using the re-diagonalized basis of BMN operators derived in [13] take the form (2.3) (at least to the first non-trivial order in λ'). In what follows, we will restrict ourselves to BMN operators with scalar impurities only and always assume the re-diagonalized basis of [13], so that (2.4) holds.

As we mentioned earlier, anomalous dimensions δ_K of non-BMN operators become infinite in the double scaling limit, and hence they do not appear in the OPE. As a result, the sum in (2.4) is reduced to BMN operators only,

$$\mathcal{O}_1(0)\mathcal{O}_2(x) = \frac{1}{|x|^{\Delta_1+\Delta_2-J}}\sum_{K\in\mathrm{BMN}} C_{12K}\,|x|^{\Delta_K-J}\,\mathcal{O}_K(0)\,,\qquad x\to 0\,. \tag{2.5}$$

Here $J = |J_1 \pm J_2|$ depending on whether $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same or opposite sign of R-charge. Note that $\Delta_K - J = n_K + \delta_K$, where n_K is the total engineering dimension of the impurities[4] in the operator $\mathcal{O}_K$ and δ_K is the anomalous dimension. In the double scaling limit δ_K is finite only for the BMN operators, hence only BMN operators contribute to the sum in (2.5) since x is small.

To make the statement (2.5) more precise, let us consider a general n-point function $\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\prod_i A_i(y_i)\rangle$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are BMN operators, and A_i are some local

[4]Each scalar impurity contributes 1; each fermion impurity contributes 3/2 and each derivative impurity contributes 2.

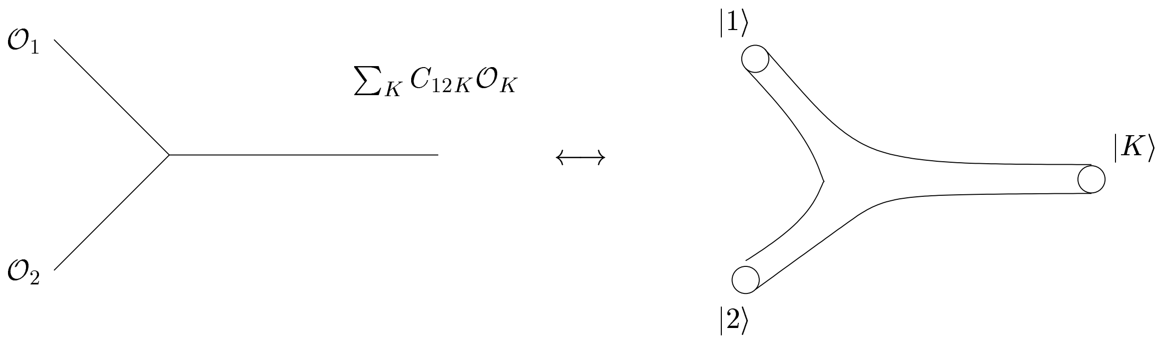

Figure 1: Short OPE of BMN operators and its string interpretation.

operators. Using (2.4) with fixed $|y_i| \gg |x|$, we obtain

$$\begin{aligned}\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\prod_i A_i(y_i)\rangle &= \sum_K \frac{C_{12K}}{|x|^{\Delta_1+\Delta_2-\Delta_K}}\,\langle \mathcal{O}_K(0)\prod_i A_i(y_i)\rangle \\ &= \frac{1}{|x|^{\Delta_1+\Delta_2-J}}\sum_{K\in\mathrm{BMN}} C_{12K}\left|\frac{x}{y_1}\right|^{\Delta_K-J} f_K(y_i) \end{aligned} \tag{2.6}$$

where we have defined $\langle \mathcal{O}_K(0)\prod_i A_i(y_i)\rangle := |y_1|^{-(\Delta_K-J)} f_K(y_i)$. In the last line of (2.6), we have used $|x/y_1| \ll 1$ and the fact that $\Delta_K - J \to \infty$ unless $\mathcal{O}_K$ is a BMN operator. This demonstrates the shortening of the OPE of the BMN operators in the double scaling limit.

The short OPE has a natural interpretation in string theory. To see this, let us use (1.5) and rewrite (2.5) in the form

$$|x|^{\Delta_1+\Delta_2-J}\mathcal{O}_1(0)\mathcal{O}_2(x) = \sum_{K\in\mathrm{BMN}} C_{12K}\,|x|^{E_K/\mu}\,\mathcal{O}_K(0)\,, \qquad x\to 0\,. \tag{2.7}$$

Here we have introduced the notation $E_K/\mu := \Delta_K - J = n_K + \delta_K$, cf. eq. (1.5). The factor C_{12K} corresponds to the 3-string interaction vertex [4], see (1.9). The factor $|x|^{E_K/\mu}$ corresponds to, apart from an overall measure factor, the integrand of the string propagator (without ghosts)

$$\frac{1}{L_0+\bar{L}_0-2} = \int \frac{d^2q}{|q|^2}\, q^{L_0-1}\,\bar{q}^{\bar{L}_0-1}\,, \tag{2.8}$$

with modulus $|q|^2$ mapped to $|x|^{1/\mu}$. Equation (2.7) has a suggestive diagrammatic representation, figure 1. We remark that the correspondence in figure 1 relies on the fact that non-BMN operators do not appear in the OPE (2.7). We will use this fundamental relation to uncover the higher point string interactions from the short distance limits of the BMN correlators.

3. n-string interactions from BMN correlators

In this section, we study correlators of BMN operators and their relation to pp-wave string interactions. For each string interaction process, we will divide the set of states into incoming and outgoing, according to whether they have negative or positive values

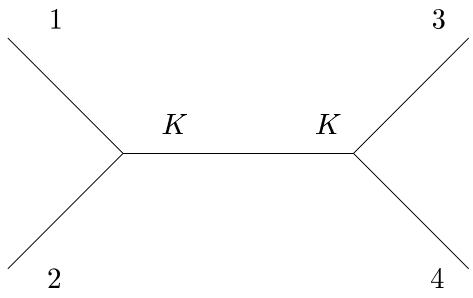

Figure 2: A four-string interaction.

of p^+. We take the convention that incoming states are associated with BMN operators made out of $\bar{Z}$ and outgoing states are associated with BMN operators made out of Z. In the analysis below, we will denote incoming operator by $\bar{\mathcal{O}}$ and outgoing operator by $\mathcal{O}$.

It is well known that the form of three-point functions in $\mathcal{N} = 4$ SYM is uniquely determined by conformal invariance. Hence, it is natural to expect that the x-independent coefficient $C_{I_1 I_2 I_3}$ of the three-point function is directly related to the three-string interaction [19], describing the joining and splitting of closed strings. The analysis carried out in [4] and more recently in [6] confirms that this is indeed the case. On the other hand, general n-point functions ($n > 3$) have a non-trivial space-time dependence and their form is not fully determined by conformal invariance. A question then arises of what is the meaning of these n-point functions of BMN operators on the string theory side. In this section we will argue that the short OPE introduced in the last section leads to a natural correspondence between the short distance limits of multi-BMN correlators and higher string interactions.

We first consider a four-string process $1 + 2 \to 3 + 4$. This corresponds to the four-point correlation function $\langle \bar{\mathcal{O}}_1(x_1)\bar{\mathcal{O}}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$ of BMN operators. The s-channel of the string process corresponds to a specific double OPE of this correlator, such that $x_{12} \to 0$ and $x_{34} \to 0$, as depicted in figure 2. More precisely, consider the following expression

$$
\begin{aligned}
&|x_{12}|^{\Delta_1+\Delta_2-J_s}|x_{34}|^{\Delta_3+\Delta_4-J_s}|x_{23}|^{2J_s}\langle \bar{\mathcal{O}}_1(x_1)\bar{\mathcal{O}}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \\
&\quad = \sum_{K\in \mathrm{BMN}} C_{\bar{1}\bar{2}\bar{K}}\, C_{34K} \left|\frac{x_{12}x_{34}}{x_{23}^2}\right|^{\frac{E_K}{\mu}}, \qquad J_s := J_1 + J_2 = J_3 + J_4\,.
\end{aligned} \tag{3.1}
$$

Here we took a specific *double pinching limit* of a conformally invariant expression involving the four-point correlator of BMN operators. The choice of pinching determines which operators we are expanding in the short OPE. Hence the r.h.s. of (3.1) is obtained from the short OPE of $\bar{\mathcal{O}}_1(x_1)\bar{\mathcal{O}}_2(x_2)$ and of $\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)$. The skeleton diagram of this double OPE is shown in figure 2 and it corresponds to the s-channel of the string process. In the expression above,

$$
J_s = J_1 + J_2 = J_3 + J_4 \tag{3.2}
$$

is the conserved R charge flowing through the s-channel. As before, $C_{\bar{1}\bar{2}\bar{K}}$ and C_{34K} correspond to two 3-string interaction vertices, and[5] $(|x_{12}||x_{34}|/|x_{23}|^2)^{\delta_K}$ corresponds, apart from an overall measure factor, to the integrand of the string propagator. The double pinching limit in (3.1) is understood as a power-series expansion in the small quantity $|x_{12}||x_{34}|/|x_{23}|^2$. In other words, only finite values of δ_K are left in the sum. The sum over the BMN operators in (3.1) corresponds precisely to the sum over *physical* intermediate string states in the s-channel.

The two other channels of the four-point string process similarly arise from the remaining two double pinching limits of the four-point BMN correlator:

$$\text{t-channel}: \lim_{x_{13},x_{24}\to 0} |x_{13}|^{\Delta_1+\Delta_3-J_t}|x_{24}|^{\Delta_2+\Delta_4-J_t}|x_{12}|^{2J_t} \times \langle \bar{\mathcal{O}}_1(x_1)\bar{\mathcal{O}}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \tag{3.3}$$

$$\text{u-channel}: \lim_{x_{14},x_{23}\to 0} |x_{14}|^{\Delta_1+\Delta_4-J_u}|x_{23}|^{\Delta_2+\Delta_3-J_u}|x_{12}|^{2J_u} \times \langle \bar{\mathcal{O}}_1(x_1)\bar{\mathcal{O}}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\,. \tag{3.4}$$

Here

$$J_t := |J_1 - J_3| = |J_2 - J_4| \qquad \text{and} \quad J_u := |J_1 - J_4| = |J_2 - J_3|\,. \tag{3.5}$$

The double pinching limits in the equations above are to be interpreted as before.

A general comment is in order. To use the OPE we have to assume the short-distance pinching limit first. After this we take the double-scaling limit (1.3) and this restricts the operators appearing on the r.h.s. of the OPE to the BMN operators. In [13, section 2.3] it was argued that the radiative corrections to the four-point correlators become infinite when the BMN limit is taken *before* the pinching limit. This does not apply to our case. In addition, the discontinuous values of the four-point functions reported in [13, section 2.2] precisely correspond to the three different channels of the four-string process.

So far we have been concerned with reproducing the integrand of the string interaction in different channels. To get the full string interaction, one has to integrate over the string moduli and sum over the different channels in string field theory. Our method, based on the analysis of short distance limits of BMN correlators, does not give information about the integration region over the string moduli. It would be interesting to understand better how and if this arises from the SYM point of view.

We now discuss the generalization to higher-point string interaction. The general approach is similar to the analysis above, but there is an important novel feature — a *hierarchy of pinchings*. To illustrate this we consider an example of seven-point string interaction $1 + 2 + 3 + 4 \to 5 + 6 + 7$ as depicted in figure 3. As it should be clear from the structure of this string interaction, the corresponding field theory correlator should be taken in the short-distance limit involving the pinching of x_{12}, x_{34}, $x_{56} \to 0$. This has to

[5]Note that $|x_{12}||x_{34}|/|x_{23}|^2$ is a conformally invariant cross-ratio in the double pinching limit.

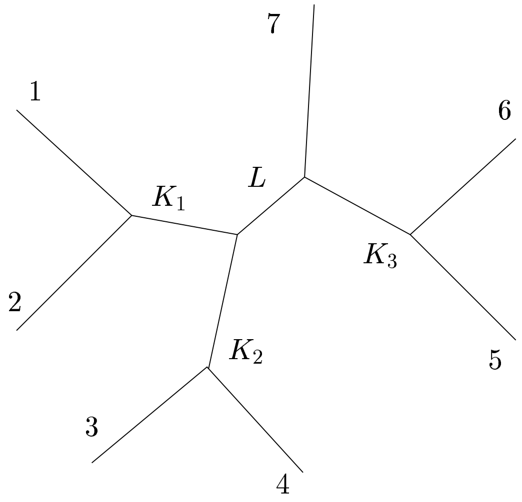

Figure 3: A seven-string interaction.

be followed by a second pinching of x_{13}, $x_{57} \to 0$. More specifically we consider

$$
\begin{aligned}
&|x_{12}|^{\Delta_1+\Delta_2-J_{12}}|x_{34}|^{\Delta_3+\Delta_4-J_{34}}|x_{56}|^{\Delta_5+\Delta_6-J_{56}}|x_{57}|^{\Delta_7-J_7}|x_{15}|^{2J}\left\langle\prod_{I=1}^{4}\bar{\mathcal{O}}_I(x_I)\prod_{I=5}^{7}\mathcal{O}_I(x_I)\right\rangle \\
&\xrightarrow{(1)} |x_{57}|^{E_7/\mu}|x_{15}|^{2J}\sum_{K_1,K_2,K_3} C_{\bar{1}\bar{2}\bar{K}_1}C_{\bar{3}\bar{4}\bar{K}_2}C_{56K_3}|x_{12}|^{\frac{E_{K_1}}{\mu}}|x_{34}|^{\frac{E_{K_2}}{\mu}}|x_{56}|^{\frac{E_{K_3}}{\mu}}\cdot \\
&\quad\cdot\langle\bar{\mathcal{O}}_{K_1}(x_1)\bar{\mathcal{O}}_{K_2}(x_3)\mathcal{O}_{K_3}(x_5)\mathcal{O}_7(x_7)\rangle \\
&\xrightarrow{(2)} \sum_{K_1,K_2,K_3,L} C_{\bar{1}\bar{2}\bar{K}_1}C_{\bar{3}\bar{4}\bar{K}_2}C_{56K_3}C_{\bar{K}_1\bar{K}_2\bar{L}}C_{K_37L}\left|\frac{x_{12}}{x_{13}}\right|^{\frac{E_{K_1}}{\mu}}\left|\frac{x_{34}}{x_{13}}\right|^{\frac{E_{K_2}}{\mu}}\left|\frac{x_{56}}{x_{57}}\right|^{\frac{E_{K_3}}{\mu}}\left|\frac{x_{13}x_{57}}{x_{15}^2}\right|^{\frac{E_L}{\mu}},
\end{aligned}
\tag{3.6}
$$

where the limit *(1)* denotes the first hierarchical pinching $|x_{12}|$, $|x_{34}|$, $|x_{56}| \to 0$, and the limit *(2)* denotes the second pinching $|x_{13}|$, $|x_{57}| \to 0$. In the above expression,

$$J_{12} := J_1 + J_2 \qquad \text{etc.,} \tag{3.7}$$

and J is the total R charge

$$J := J_1 + J_2 + J_3 + J_4 \tag{3.8}$$

As before, the three-point coefficients C_{IJK} correspond to the three-string vertices, and the x-ratios are identified with the moduli of the corresponding string propagators. The sum over all such inequivalent skeleton diagrams in field theory, appropriately integrated over, corresponds to the full string interaction.

In the above, we have discussed how, in a certain short distance limit (which involves a hierarchy of multi-pinchings) an n-point BMN correlator corresponds to a tree n-string interaction in a specific channel. Now we discuss how to obtain string loop interactions from the BMN correlators. The analysis again hinges on the use of the short OPE, but there is yet another important new feature of the procedure — cluster decomposition. To

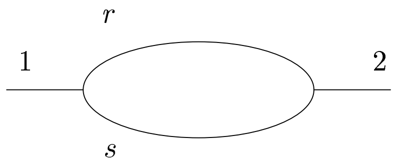

Figure 4: A one-loop corrected string interaction.

illustrate the idea, it is sufficient to consider the simple case of a one-loop two-point string interaction drawn in figure 4. To reproduce this two-point amplitude, we have to consider a *six*-point BMN correlator.

Consider the following quantity I and its limits

$$\begin{aligned}
I \;&:= \frac{|x_{12}|^{2\Delta_1}|x_{46}|^{2\Delta_2}}{x_{23}^{\Delta_1}x_{45}^{\Delta_2}} \\
&\quad\times \sum_{r,s\in \mathrm{BMN}} \langle \mathcal{O}_1(x_1)\bar{\mathcal{O}}_r(x_2)\bar{\mathcal{O}}_s(x_3)\mathcal{O}_r(x_4)\mathcal{O}_s(x_5)\bar{\mathcal{O}}_2(x_6)\rangle |x_{23}|^{2\Delta_r+\Delta_s-J_r}|x_{45}|^{2\Delta_s+\Delta_r-J_s} \\
&\xrightarrow{(1)} \frac{|x_{12}|^{2\Delta_1}|x_{46}|^{2\Delta_2}}{x_{23}^{\Delta_1}x_{45}^{\Delta_2}} \sum_{r,s,a,b\in \mathrm{BMN}} C_{\bar{r}\bar{s}\bar{a}}C_{rsb}|x_{23}|^{\frac{E_r}{\mu}+\Delta_a}|x_{45}|^{\frac{E_s}{\mu}+\Delta_b}\langle \mathcal{O}_1(x_1)\bar{\mathcal{O}}_a(x_2)\mathcal{O}_b(x_4)\bar{\mathcal{O}}_2(x_6)\rangle \\
&\xrightarrow{(2)} \sum_{r,s\in \mathrm{BMN}} C_{\bar{r}\bar{s}\bar{1}}C_{rs2}\,|x_{23}|^{\frac{E_r}{\mu}}|x_{45}|^{\frac{E_s}{\mu}} \qquad (3.9)
\end{aligned}$$

Here the limit *(1)* denotes the short distance limit $|x_{23}| \to 0$, $|x_{45}| \to 0$. After this limit, I becomes a function of x_1, x_2 and x_4, x_6. The second limit *(2)* is a *large distance cluster limit* where we group x_1, x_2 and x_4, x_6 into two independent clusters and send them far away from each other. Due to the cluster decomposition principle, which holds in a general QFT, the four-point function in the second line of (3.9) factorizes in this limit as

$$\langle \mathcal{O}_1(x_1)\bar{\mathcal{O}}_a(x_2)\mathcal{O}_b(x_4)\bar{\mathcal{O}}_2(x_6)\rangle = \frac{\delta_{1a}}{|x_{12}|^{2\Delta_1}}\frac{\delta_{2b}}{|x_{46}|^{2\Delta_2}}\,, \qquad (3.10)$$

and the last line in (3.9) follows. As before, the three-point coefficients C_{IJK} correspond to the three-string vertices. x_{23} and x_{45} are identified with the moduli of the corresponding string propagators. Note that only physical degree of freedom propagate in the string loop in the lightcone gauge. The last line of (3.9), when appropriately integrated over, corresponds to the full one-loop two-point string interaction.

It should be clear from the above analysis how to generalize to the higher loop case. Generally, to obtain an n-point h-loop string interaction for a specific string field theory diagram, one has to start with a $3v$-point BMN correlator where v is the number of vertices in the string diagram. Short distance limit (like *(1)* above) generates the string propagators. Then it is followed by a large distance clustering limit which separates the v vertices. The resulting expression is in direct correspondence with the string loop interaction.

Note that in the pinching limit we proposed above, a general string interaction is effectively reduced to *two-point correlators* in field theory. This appears to be in agreement with a recent proposal of Verlinde [11] which states that pp-wave string interactions should be extracted from two-point correlators in field theory.

4. On instanton corrections to BMN correlators

Due to the very nature of the Penrose limit, which relies on the existence of null geodesics, the pp-wave metric cannot be euclideanized and stay real. This suggests that there are no D-instantons in a pp-wave background. For the pp-wave/SYM correspondence to hold, this means that there should be no Yang-Mills instanton corrections to the SYM quantities which are relevant for the correspondence. A priori, there is no reason to expect that instantons do not contribute to generic SYM correlators since $g^2_{\rm YM}$ is fixed in the double scaling limit (1.3).

We start by examining two-point functions. Our analysis is similar to that in [20], which showed that instanton corrections to extremal correlators in $\mathcal{N}=4$ SYM vanish. Recall that in the case of $\mathcal{N}=4$ SYM with gauge group SU(N), the Dirac operator in the adjoint representation in the background of an instanton of winding number k has $8kN$ zero modes. However, only 16 of them are exact zero modes, since the remaining ones are lifted by the presence of a fermion quadrilinear, which is induced by the Yukawa term in the instanton action [21]. All the considerations in this section apply to instantons of arbitrary charge. For full details of the ADHM instanton calculus we refer the reader to the review [22]. The exact zero modes can be generated acting with supersymmetry and superconformal transformation on the instanton, and take the form

$$\lambda^I(x) = \frac{1}{2} F_{\mu\nu}\sigma^{\mu\nu}\zeta^I(x)\,. \tag{4.1}$$

Here $\zeta^I(x) = \xi^I + \bar\eta^I\bar\sigma_\mu(x-x_0)^\mu$ and $I=(0,i)$, with $i=1,2,3$. ξ^I, $\bar\eta^I$ are eight constant Weyl spinors, and x_0 is the centre of the instanton configuration. Solving the equations of motion for the three complex scalar fields ϕ^i (and $\bar\phi_i$) of $\mathcal{N}=4$ SYM, we obtain

$$\phi^i_{\rm cl} = \frac{1}{2}\,\zeta^0 F_{\mu\nu}\sigma^{\mu\nu}\zeta^i\,, \tag{4.2}$$

$$\phi^\dagger_{i\,\rm cl} = \epsilon_{ijk}\,\zeta^j F_{\mu\nu}\sigma^{\mu\nu}\zeta^k\,. \tag{4.3}$$

We can choose for example $\phi^1_{\rm cl}$ (resp. $\phi^2_{\rm cl}$, $\phi^3_{\rm cl}$) to be the instanton components of the field Z (resp. Φ, Ψ). To calculate instanton contributions to the BMN correlators we write each scalar field in the composite BMN operator as

$$\phi^i = \phi^i_{\rm cl} + \delta\phi^i\,, \qquad \phi^\dagger_i = \phi^\dagger_{i\,\rm cl} + \delta\phi^\dagger_i\,, \tag{4.4}$$

where $\phi^i_{\rm cl}$, $\phi^\dagger_{i\,\rm cl}$ are the instanton background fields (4.2)

$$Z_{\rm cl}\sim\zeta^0\zeta^1\,,\quad \Phi_{\rm cl}\sim\zeta^0\zeta^2\,,\quad \Psi_{\rm cl}\sim\zeta^0\zeta^3,\quad \bar Z_{\rm cl}\sim\zeta^2\zeta^3\,,\quad \bar\Phi_{\rm cl}\sim\zeta^1\zeta^3\,,\quad \bar\Psi_{\rm cl}\sim\zeta^1\zeta^2\,, \tag{4.5}$$

induced by the exact fermion zero modes, and $\delta\phi^i$, $\delta\phi^\dagger_i$ represent the fluctuations of scalar fields in the instanton background (they also include contributions of the lifted fermion zero modes). Unless all the sixteen exact fermion zero modes are saturated, instanton corrections to the correlator $\langle\mathcal{O}_{\rm BMN}(x)\bar{\mathcal{O}}_{\rm BMN}(0)\rangle$ vanish. Due to the form (4.5) and since $(\zeta(x))^n=0$ for $n\geq 3$, it is obvious that $\mathcal{O}_{\rm BMN}$ and $\bar{\mathcal{O}}_{\rm BMN}$ must each provide eight fermion

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zero modes $(\zeta^0)^2(\zeta^1)^2(\zeta^2)^2(\zeta^3)^2$ in order to be able to saturate the sixteen exact fermion zero modes. Since we need two powers of ζ^0 and since ζ^0 enters only the fields Z, Φ, Ψ, potentially non-vanishing contributions can come only from

$$\mathcal{O}_{\rm BMN} = (Z\Phi \text{ or } Z\Psi \text{ or } \Phi\Psi \text{ or } ZZ \text{ or } \Phi\Phi \text{ or } \Phi\Psi)_{\rm cl} \times (2 \text{ more } \phi^\dagger_{\rm cl}) \times (\ \delta\phi\text{'s and } \delta\phi^\dagger\text{'s}). \quad (4.6)$$

Note that $\mathcal{O}_{\rm BMN}$ cannot contain Φ and $\bar{\Phi}$, Ψ and $\bar{\Psi}$ or Z and $\bar{Z}$ simultaneously (however "non-holomorphic" BMN operators which contain Φ and $\bar{\Psi}$ are allowed). Using the explicit form (4.5), one can check that (4.6) can never generate the required combination $(\zeta^0)^2(\zeta^1)^2(\zeta^2)^2(\zeta^3)^2$. Similarly, one can show that $\bar{\mathcal{O}}_{\rm BMN}$ can never give rise to the zero mode structure $(\zeta^0)^2(\zeta^1)^2(\zeta^2)^2(\zeta^3)^2$. Hence we conclude that there are no instanton contributions to the two-point functions of BMN operators.

Next we consider three-point functions. Using the above analysis, it is clear that to saturate all sixteen zero modes, two of the operators must provide six zero modes each, and the remaining one – four zero modes. For example, consider the correlator

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\mathcal{O}_3(0)\rangle \quad (4.7)$$

where schematically (i.e. discarding the phase factors and the sums)

$$\mathcal{O}_1 := {\rm tr}[Z^{J_1}\Phi\bar{\Psi}]\,, \qquad \mathcal{O}_2 := {\rm tr}[Z^{J_2}\bar{\Phi}\Psi]\,, \qquad \mathcal{O}_3 := {\rm tr}[\bar{Z}^J]\,, \quad (4.8)$$

with $J = J_1 + J_2$. Now it is easy to see that all sixteen fermion zero modes can be saturated since

$$\mathcal{O}_1 \ni (\delta Z)^{J_1-1}(Z\Phi\bar{\Psi})_{\rm cl} \sim (\zeta^0)^2(\zeta^1)^2(\zeta^2)^2\,, \quad (4.9)$$
$$\mathcal{O}_2 \ni (\delta Z)^{J_2-1}(Z\bar{\Phi}\Psi)_{\rm cl} \sim (\zeta^0)^2(\zeta^1)^2(\zeta^3)^2\,, \quad (4.10)$$
$$\mathcal{O}_3 \ni (\delta\bar{Z})^{J-2}(\bar{Z}^2)_{\rm cl} \sim (\zeta^2)^2(\zeta^3)^2\,. \quad (4.11)$$

We conclude with a few remarks.

1. We saw that our simple argument based on fermion zero mode counting does not work for three-point functions of non-holomorphic BMN operators. To see if the instanton contribution to these correlators vanishes one would require a more detailed calculation which would have to take into account the precise form of the BMN operators — including the phase factors, the sums and the specific choice of a re-diagonalized basis.

2. Instanton contributions to three-point functions of holomorphic BMN operators with scalar impurities vanishes automatically.

3. We checked that when the BMN phase factors are absent, the instanton contributions to all three-point functions vanish due to a precise term by term cancellation. This was expected since in this case BMN operators are protected, and their two- and three-point functions do not receive either perturbative or nonperturbative corrections [23, 24].

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4. Finally, instanton corrections to all BMN two-point functions vanish automatically and this indicates that the two-point functions are the preferred building blocks of the pp-wave/SYM dictionary. This is in agreement with discussions in the recent literature.

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